Sub: Mathematics (General) — B.Sc. 4th Semester

Differentiation of Vectors

Scalar function:- Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique real number $f(t)$, then this rule defines a scalar function of the scalar variable t. Here $f(t)$ is a scalar quantity and thus f is a scalar function.

Vector function:- Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique vector $\vec{f}(t)$, then this rule defines a vector function of the scalar variable t. Here $\vec{f}(t)$ is a vector quantity and thus $\vec{f}$ is a vector function.

$$\therefore \vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$ where $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar unit vectors

Again $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\vec{f}(t) = \vec{r}$$

$$\Rightarrow f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow f_1(t) = x, \quad f_2(t) = y \text{ and } f_3(t) = z$$

For example: If $\vec{f}(t) = \vec{r}$ and

(1) $\vec{f}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + 0k$ and

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$\vec{f}(t) = \vec{r} \Rightarrow x = a\cos t, \ y = a\sin t$

$\therefore x^2 + y^2 = a^2$ which represent a circle.

(2) If $\vec{f}(t) = (a\cos t)\hat{i} + (b\sin t)\hat{j} + 0k$ then

$x = a\cos t, \quad y = b\sin t,$

$\therefore \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$, which represent an ellipse.

Limit and Continuity of a vector function:-

Def^n 1:- A vector function $\vec{f}(t)$ is said to tend to a limit $\vec{l}$ when t tends to t_0, if for any given positive number ϵ, however small, there corresponds a positive number δ such that $|\vec{f}(t) - \vec{l}| < \epsilon$ whenever $0 < |t - t_0| < \delta$

If $\vec{f}(t)$ tends to a limit $\vec{l}$ as t tends to t_0, we write $\lim_{t \to t_0} \vec{f}(t) = \vec{l}$.

If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ then

$$\lim_{t \to t_0} \vec{f}(t) = \lim_{t \to t_0} f_1(t)\hat{i} + \lim_{t \to t_0} f_2(t)\hat{j} + \lim_{t \to t_0} f_3(t)\hat{k}$$

For example:- If $\vec{f}(t) = (a\cos t)\hat{i} + (b\sin t)\hat{j} + (c\tan t)\hat{k}$

then

$$\lim_{t \to \pi/4} \vec{f}(t) = \lim_{t \to \pi/4} (a\cos t)\hat{i} + \lim_{t \to \pi/4} (b\sin t)\hat{j} + \lim_{t \to \pi/4} (c\tan t)\hat{k}$$

$$= (a\cos\pi/4)\hat{i} + (b\sin\pi/4)\hat{j} + (c\tan\pi/4)\hat{k}$$

$$= \frac{1}{\sqrt{2}} a\hat{i} + \frac{1}{\sqrt{2}} b\hat{j} + c\hat{k} \;//$$

Def^n 2:- A vector function $\vec{f}(t)$ is said to be continuous for a value t_0 of t if

(i) $\vec{f}(t_0)$ is defined

(ii) for any given positive number ϵ, however small, there corresponds a positive number δ such that $|\vec{f}(t) - \vec{f}(t_0)| < \epsilon$ whenever $|t - t_0| < \delta$

Theorem 1:- The necessary and sufficient condition for a vector function $\vec{f}(t)$ to be continuous at $t = t_0$ is that

$$\lim_{t \to t_0} \vec{f}(t) = \vec{f}(t_0)$$

Theorem 2:- If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ then $\vec{f}(t)$ is continuous iff $f_1(t), f_2(t), f_3(t)$ are continuous (scalar functions)

Th^m 3:- Let $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$
and $\vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$

Then the necessary and sufficient conditions that $\lim_{t\to t_0} \vec{f}(t) = \vec{l}$ are $\lim_{t\to t_0} f_1(t) = l_1$, $\lim_{t\to t_0} f_2(t) = l_2$ and $\lim_{t\to t_0} f_3(t) = l_3$

Th^m 4:- If $\vec{f}(t), \vec{g}(t)$ are vector function of scalar variable t and $\phi(t)$ is a scalar function of scalar variable t, then

(i) $\lim_{t\to t_0} [\vec{f}(t) \pm \vec{g}(t)] = \lim_{t\to t_0} \vec{f}(t) \pm \lim_{t\to t_0} \vec{g}(t)$

(ii) $\lim_{t\to t_0} [\vec{f}(t) \cdot \vec{g}(t)] = [\lim_{t\to t_0} \vec{f}(t)] \cdot [\lim_{t\to t_0} \vec{g}(t)]$

(iii) $\lim_{t\to t_0} [\vec{f}(t) \times \vec{g}(t)] = [\lim_{t\to t_0} \vec{f}(t)] \times [\lim_{t\to t_0} \vec{g}(t)]$

(iv) $\lim_{t\to t_0} [\phi(t) \vec{f}(t)] = [\lim_{t\to t_0} \phi(t)] [\lim_{t\to t_0} \vec{f}(t)]$

(v) $\lim_{t\to t_0} |\vec{f}(t)| = |\lim_{t\to t_0} \vec{f}(t)|$

Derivative of a vector function with respect to a scalar:-

Def^n:- Let $\vec{r} = \vec{f}(t)$ be a vector function of the scalar variable t. We define $\vec{r} + \delta\vec{r} = \vec{f}(t+\delta t)$

$$\therefore \delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)$$

Consider the vector $\dfrac{\delta\vec{r}}{\delta t} = \dfrac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$

If $\lim_{\delta t\to 0} \dfrac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0} \dfrac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$ exists, then the value of this limit is called the derivative or differential coefficient of $\vec{r}$ w.r.t. the scalar t and it is denoted by $\dfrac{d\vec{r}}{dt}$.

Successive Derivatives:- If $\vec{r}$ is a vector function of the scalar variable t, then $\frac{d\vec{r}}{dt}$ is also in general a vector function of t. If $\frac{d\vec{r}}{dt}$ is differentiable, then its derivative is denoted by $\frac{d^2\vec{r}}{dt^2}$ and is called the second order derivatives of $\vec{r}$ and so on. $\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \ldots$ are also represented by $\dot{\vec{r}}, \ddot{\vec{r}}, \ldots$ respectively.

Derivatives in terms of components :-

Let $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

$\therefore \vec{f}(t+\delta t) = f_1(t+\delta t)\hat{i} + f_2(t+\delta t)\hat{j} + f_3(t+\delta t)\hat{k}$

$$\begin{aligned}\therefore \delta\vec{f} &= \vec{f}(t+\delta t) - \vec{f}(t)\\ &= [f_1(t+\delta t) - f_1(t)]\hat{i} + [f_2(t+\delta t) - f_2(t)]\hat{j} + [f_3(t+\delta t) - f_3(t)]\hat{k}\\ &= \delta f_1\hat{i} + \delta f_2\hat{j} + \delta f_3\hat{k} \text{ (say)}\end{aligned}$$

and $\dfrac{\delta\vec{f}}{\delta t} = \dfrac{\delta f_1}{\delta t}\hat{i} + \dfrac{\delta f_2}{\delta t}\hat{j} + \dfrac{\delta f_3}{\delta t}\hat{k}$

When $\delta t \to 0$ we get

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Therefore, to differentiate a vector function, we are to differentiate the scalar components

Differentiation Formulae

Thm: If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are differentiable vector function of a scalar variable t and ϕ is a differentiable scalar function of the same variable t, then

(1) $\dfrac{d}{dt}(\vec{a} \pm \vec{b}) = \dfrac{d\vec{a}}{dt} \pm \dfrac{d\vec{b}}{dt}$

(2) $\dfrac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\cdot\vec{b}$

(3) $\dfrac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\times\vec{b}$

(4) $\dfrac{d}{dt}(\phi\vec{a}) = \phi\dfrac{d\vec{a}}{dt} + \dfrac{d\phi}{dt}\vec{a}$

(5) $\dfrac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\dfrac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \dfrac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \dfrac{d\vec{c}}{dt}\right]$

(6) $\dfrac{d}{dt}\{\vec{a}\times(\vec{b}\times\vec{c})\} = \dfrac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left(\dfrac{d\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\dfrac{d\vec{c}}{dt}\right)$

Derivative of a constant vector :- A vector is said to be constant only if both its magnitude and direction are fixed.

Let $\vec{f}(t)$ be a constant vector function of the scalar variable t. Let $\vec{f}(t) = \vec{c}$, where $\vec{c}$ is a constant vector

Then $\vec{f}(t+\delta t) = \vec{c}$

$\Rightarrow \vec{f}(t) + \delta\vec{f}(t) = \vec{c}$

$\Rightarrow \delta\vec{f}(t) = \vec{0}$ (zero vector)

$\Rightarrow \lim_{\delta t \to 0} \frac{\delta \vec{f}}{\delta t} = \lim_{\delta t \to 0} \vec{0} = \vec{0}$

$\therefore \frac{d\vec{f}}{dt} = \vec{0}$

Thus the derivative of a constant vector is equal to the null vector.//

Thm :- 1. If $\vec{a}$ is a vector function of a scalar variable t and if $|\vec{a}| = a$ then

(i) $\frac{d}{dt}(\vec{a}^2) = 2a\frac{da}{dt}$

(ii) $\vec{a} \cdot \frac{d\vec{a}}{dt} = a\frac{da}{dt}$

Pf: (i) We have $\vec{a}^2 = \vec{a} \cdot \vec{a} = a \cdot a \cos 0 = a^2$

$\therefore \frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(a^2) = 2a\frac{da}{dt}$ //

(ii) We have $\frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(\vec{a} \cdot \vec{a})$

$= \frac{d\vec{a}}{dt} \cdot \vec{a} + \vec{a} \cdot \frac{d\vec{a}}{dt} = 2\vec{a} \cdot \frac{d\vec{a}}{dt} \rightarrow (a)$

Also $\frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(a^2) = 2a\frac{da}{dt} \rightarrow (b)$

$\therefore$ From (a) & (b) $\Rightarrow \cancel{2}\vec{a} \cdot \frac{d\vec{a}}{dt} = \cancel{2}a\frac{da}{dt}$

or, $\vec{a} \cdot \frac{d\vec{a}}{dt} = a\frac{da}{dt}$ //

Thm 2:- The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant magnitude is $\vec{a}\cdot\frac{d\vec{a}}{dt}=0$

Pf: Let $\vec{a}$ be a vector function of the scalar variable t.

Let $|\vec{a}| = a = \text{constant}$.

Then $\vec{a}\cdot\vec{a} = a^2 = \text{constant}$

$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

or, $\vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a} = 0$

or, $2\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

or, $\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

Therefore the condition is necessary.

Condition is sufficient:-

If $\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$, then

$\vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a} = 0$

or, $\frac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

or, $\vec{a}\cdot\vec{a} = \text{constant}$

or, $a^2 = \text{constant}$

or, $|\vec{a}| = \text{constant}$ //

Thm 3:- If $\vec{a}$ has constant length (fixed magnitude), then $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$

Pf. Let $|\vec{a}| = a = \text{constant}$. Then $\vec{a}\cdot\vec{a} = a^2 = \text{constant}$

$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

or, $\vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a} = 0$

or, $2\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

or, $\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

Thus the scalar product of the two vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ is zero. Therefore $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular to each other provided $\frac{d\vec{a}}{dt}$ is not null vector i.e $\left|\frac{d\vec{a}}{dt}\right| \neq 0$

Th^m 4:- The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant direction is

$$\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$$

Pf. Let $\vec{a}$ be a vector function of the scalar variable t. Let $\hat{A}$ be a unit vector in the direction of $\vec{a}$. If a be the magnitude of $\vec{a}$, then $\vec{a} = a\hat{A}$

$$\therefore \frac{d\vec{a}}{dt} = a\frac{d\hat{A}}{dt} + \frac{da}{dt}\hat{A}$$

Hence
$$\vec{a} \times \frac{d\vec{a}}{dt} = (a\hat{A}) \times \left(a\frac{d\hat{A}}{dt} + \frac{da}{dt}\hat{A}\right)$$
$$= a^2\hat{A} \times \frac{d\hat{A}}{dt} + a\frac{da}{dt}\hat{A}\times\hat{A}$$
$$= a^2\hat{A} \times \frac{d\hat{A}}{dt} \quad [\because \hat{A}\times\hat{A} = \vec{0}] \longrightarrow (1)$$

The condition is necessary:- Suppose $\vec{a}$ has a constant direction. Then $\hat{A}$ is (or $\frac{\vec{a}}{a}$) a constant vector because it has constant direction as well as constant magnitude.

Therefore $\frac{d\hat{A}}{dt} = \vec{0}$

$\therefore$ From (1) $\vec{a} \times \frac{d\vec{a}}{dt} = a^2\hat{A} \times \vec{0} = \vec{0}$

Therefore the condition is necessary.

The condition is sufficient:-

Suppose that $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$

Then from (1), we get $a^2\hat{A} \times \frac{d\hat{A}}{dt} = \vec{0}$

or $\hat{A} \times \frac{d\hat{A}}{dt} = \vec{0} \longrightarrow (2)$

Since $\hat{A}$ is of constant length, therefore

$$\hat{A} \cdot \frac{d\hat{A}}{dt} = 0 \longrightarrow (3)$$

From (2) and (3), we get $\frac{d\hat{A}}{dt} = \vec{0}$

Hence $\hat{A}$ is a constant vector i.e the direction of $\vec{a}$ is constant. //

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